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GAIN LIMITATIONS ON EQUALIZERS  
AND MATCHING NETWORKS

by

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ABSTRACT

If an equalizer amplitude response curve is specified, it will be shown that the minimum flat loss obtainable with physical networks is determined. This flat loss, or scale factor on the response curve, is a function of the equalizer output terminating impedance which is arbitrary but prescribed, and the specified tolerance on input mismatch.

If the output impedance is purely reactive, the limitations on maximum voltage transfer are obtained from a consideration of the open circuit impedance parameters of the system. If power or voltage transfer to a load with finite real part is to be optimized, the scattering parameters of the system are used to determine the limits of performance.

Examples will be given comparing the performance of matched and lossless equalizers. In many practical cases the latter do not have substantially higher gain than the matched equalizer.

## I\* Definition of Equalizer Problem

The equalizer problem considered here concerns the transfer of voltage, current, or power from a prescribed generator with resistive internal impedance to a load whose impedance is a given function of frequency. It is presumed that a real frequency function is specified which defines the shape of transfer gain characteristic desired, and it is required to find a passive linear reciprocal equalizer network (a two terminal pair transducer, or two-port) which when placed between generator and load produces the specified gain shape and does so with maximum scale factor i.e. minimum flat loss. A gain characteristic which ideally is constant over a finite frequency band and zero elsewhere will be of major interest, and many of the results given may therefore be regarded as generalizations of the concept of maximum "gain-bandwidth product".<sup>1</sup>

An additional specification is the tolerance on input mismatch. It is only because the equalizer networks investigated here are not limited to the lossless case that this specification can be set independently of the others. In most of the examples given below the extremes of a matched input dissipative equalizer (zero mismatch) and a completely lossless equalizer (but not matched) will be compared.

Various aspects of the equalizer problem have been previously considered. Bode<sup>1</sup> discusses the limitations on "gain-bandwidth" product imposed by a load with shunt capacitance when a lossless equalizer is used, and also gives some consideration to matched equalizers for voltage transfer to a reactive load<sup>2</sup>. Fano<sup>3</sup> has treated the problem of optimum match of an arbitrary load with a lossless network and La Rosa and Carlin<sup>4,5,6</sup> have examined this problem when the lossless restriction on the matching network is removed. Norde<sup>7</sup> has treated matched minimum phase voltage equalizers for reactive loads. Other work on special aspects of "gain-bandwidth" product is too extensive to be given here. Wheeler<sup>8</sup> and Hansen<sup>9</sup> are typical references.

The present paper considers the general approach to any equalization problem and stems directly from the references cited above. The results presented on optimum voltage transfer to an arbitrary load (including the purely reactive load case) have not been given elsewhere.

## II. General Approach to Equalization of an Arbitrary Load

There are two basic restrictions which govern the design of an equalizer network. One of these is the general requirement of physical realizability on the overall network which includes both equalizer and prescribed load. The other is the total set of constraints specifically imposed by the load and this should be entirely independent of the equalizer network. If these constraints are satisfied, then when the overall network is synthesized, and the given load removed, the remaining circuit (the equalizer alone) is physically realizable. The form in which these restrictions are

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\* This was presented at the IRE National Convention, Symposium on Network Equalization, March 24, 1954.

stated must be such as to explicitly (and preferably in a simple way) involve the transfer gain function whose scale factor is to be maximized. The process of finding the optimum equalization is then to adjust the gain function within these general restrictions until the limits of physical realizability are attained.

The constraints which apply to the equalization of a load containing dissipation (the purely reactive load case is considered later) are most readily obtained by representing the prescribed load over the infinite frequency spectrum as a purely reactive 2-port with fixed elements terminated in a unit resistor. (Hereafter the generator resistance will be presumed normalized to unity<sup>10,11</sup>. This representation is always possible and the single resistor is sufficient to account for all the power dissipated in the load.) The "overall network" is now defined as the equalizer plus the reactance two port portion of the load.

In order that the "overall network" be physically realizable, it must have an array of scattering coefficients<sup>12</sup>  $S_{11}(p)$ ,  $S_{22}(p)$ ,  $S_{12}(p)$  which form the matrix of a positive definite or semi-definite Hermitian form for  $\text{Re } p \geq 0$ <sup>5,6,12,13</sup>. The algebraic expression of this requirement gives the general set of realizability constraints previously referred to.

The specific load constraints are obtained from the fact that at certain real and complex frequencies no power can be transferred to the load no matter what equalizer network is used. These frequencies are the points on  $j\omega$  and in the right hand half of the  $p$  plane at which the reactive 2-port portion of the load has zeros of transmission. At these frequencies the transmission factor of the overall network  $S_{12}(p)$  must generally have a zero of transmission of order  $2n$  if the load zero is of order  $n$ . Further the reflection factor looking in at the back end of the "overall network" i.e.  $S_{22}(p)$  and generally its first  $2n-1$  derivatives are completely determined by the reactance 2-port portion of the load. These properties follow from a consideration of the scattering equations for the cascade connection of a pair of two-ports<sup>3</sup>, and constitute the "load constraints" referred to earlier. It must be emphasized that load constraints are independent of the equalizer.

The load constraints amount to the statement that essentially the first  $2n$  Taylor coefficients of the back end reflection factor of both load and overall network are equal in the series expansion about a load zero of transmission.

These requirements may be expressed in terms of the Cauchy formulas for the Taylor coefficients, and as a final result one obtains integral formulas for the logarithm of the amplitude of the back end reflection factor. If the overall network is specified so that it satisfies the general realizability requirements and in addition meets the limits on  $\ln |1/S_{22}(j\omega)|$  imposed by the integral formulas, then the prescribed load can always be separated from the overall network leaving a physically realizable equalizer 2-port. A statement of these realizability conditions in the form of a theorem essentially as

given by La Rosa<sup>4,5</sup> is as follows:

### Theorem 1

The necessary and sufficient conditions that a scattering matrix  $\bar{S}(p)$  ( $p = \sigma + j\omega$ ) represent an overall network composed of an equalizer in tandem with a prescribed lossless 2-port (the reactance 2-port portion of the prescribed load) is:

(a) Matrix  $\bar{S}$  should be realizable i.e.  $I - \bar{S}(j\omega)^T \bar{S}(j\omega)$  must be the matrix of a positive definite or semi-definite hermitian form, with  $\bar{S}$  symmetric and its elements rational functions of  $p$  with real coefficients, and no right half plane poles.<sup>13</sup>

(b) Right hand and boundary zeros of transmission of the load must appear in the transmission factor  $S_{12}(p)$  of the overall network with at least the same multiplicity.

(c)\* A set of integral restrictions on  $\ln |1/S_{22}(j\omega)|$  of the form

$$\int_0^{\infty} f_1(\omega) \ln \left| \frac{1}{S_{22}(j\omega)} \right| d\omega = K_1 \quad (1)$$

must be simultaneously satisfied at all the zeros of transmission of the load. Each  $n$ th order zero contributes  $N_1$  integral equations with

$$N_1 = \begin{cases} n & \text{for a zero at zero or infinity} \\ 2n & \text{for a zero on } j\omega \\ 2n - n_0 & \text{for a right hand zero on the real axis} \\ 4n - 2n_0 & \text{for a conjugate pair of zeros in the right half plane.} \end{cases}$$

$n_0$  is the order of any right half plane zero of load transmission coincident with a zero of back end load reflection factor, i.e. of the reactance 2-port portion of the load.

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This theorem can be applied in a direct and simple fashion to a variety of equalization problems involving a load containing dissipative elements. Special consideration will be required for problems involving a purely reactive load.

### III. Power Transfer Equalization

The application of the theorem given in the preceding section requires a determination of the relationships between the transfer function

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\*The weighting functions  $f_1(\omega)$  are tabulated by Fano<sup>3</sup>. The  $K_1$  are related to the Taylor coefficients of the load at the zeros of transmission and are also tabulated in the same reference.



which is to be optimized and the reflection factor amplitude of the overall network  $|S_{22}(j\omega)|$ . The integral equations can then be used to determine "gain-bandwidth" type of restrictions on the equalizer. Part (a) of the theorem contains the necessary information for relating the reflection factor function  $S_{22}(j\omega)$  to the equalization response of the overall network. In the case of power transfer from a generator with unit internal impedance to a load (represented in Darlington form), the insertion power gain at real frequencies of the overall network normalized to the available generator power is

$$\frac{P_L}{P_0} = |S_{12}(j\omega)|^2 \quad (2)$$

$P_L$  is the power delivered to the load,  $P_0$  is the available generator power ( $|V_g|^2/4$ , where  $V_g$  is the generator open-circuit voltage) and  $|S_{12}(j\omega)|$  is the amplitude of the voltage transmission coefficient of the overall network. Since this power transfer function is an element of the scattering matrix, it is directly related to  $|S_{22}(j\omega)|$  by the general realizability constraints of part (a) of the theorem. La Rosa<sup>5,6</sup> has shown that this portion of the theorem leads to the following necessary requirement on  $|S_{12}(j\omega)|$  for an equalizer which maximizes the scale factor of power function when the shape is specified.

$$|S_{12}(j\omega)|^2 = (1 - |S_{22}(j\omega)|)(1 + |S_{11}(j\omega)|)|S_{22}(j\omega)| \geq |S_{11}(j\omega)| \quad (3)$$

In equation (3),  $|S_{11}(j\omega)|$  is a specified input reflection factor amplitude function which sets the tolerance on input mismatch. In the special case that the equalizer is lossless,  $|S_{11}(j\omega)| = |S_{22}(j\omega)|$  and equation (3) becomes

$$|S_{12}(j\omega)|^2 = 1 - |S_{22}(j\omega)|^2 \quad (4)$$

(lossless equalizer)

In another special case where the input mismatch is zero i.e.  $S_{11}(p) = 0$  equation (3) reduces to

$$|S_{12}(j\omega)|^2 = 1 - |S_{22}(j\omega)| \quad (5)$$

(matched equalizer)

Equations (3), (4), (5) give the desired relations between  $|S_{22}(j\omega)|$  and the power transfer function  $|S_{12}(j\omega)|^2$ . The integral equations, (1), may therefore be expressed in terms of the power transfer function and solved to obtain the maximum scale factor. The details of this procedure as well as examples are given in reference<sup>4,5,6</sup>. The solution of the equations always gives a unique maximum-gain scale factor for a prescribed shape of power transfer function and this cannot be exceeded by any physical equalizer. In the case where the equalizer is to produce a flat pass band with zero gain outside this band, the solution for the scale factor is particularly simple and is found directly in terms of a minimum constant value of  $|S_{22}(j\omega)| = |S_{22}|$  over the prescribed band with  $|S_{22}(j\omega)| = 1$  elsewhere. In this case it is interesting to compare the optimum lossless and matched equalizers using

equations (4) and (5)

$$10 \log \frac{|S_{12}|^2 \text{ (lossless)}}{|S_{12}|^2 \text{ (matched)}} = 10 \log (1 + |S_{22}|) \leq 3 \text{ db} \quad (6)$$

since  $|S_{22}| \leq 1$ . In any practical design of a flat power equalizer  $|S_{22}|$  is considerably less than one so that over a specified band the gain of an optimum matched design is much closer to the gain of an optimum lossless equalizer than the outside limit of 3 db given by equation (6).

#### IV. Voltage Transfer Equalization of General Dissipative Load

##### A. Integral Constraints for Voltage Transfer

The theorem given in section 2 may be applied to the problem of voltage equalization provided the voltage transfer function can be related to the scattering coefficients of the overall network. The voltage transfer function at real frequencies is taken to be

$$e = \left| \frac{V_2}{V_1} \right| \quad (7)$$

where  $|V_2|$  is the amplitude of the voltage appearing across the load and  $|V_1|$  is the open circuit (fixed) voltage amplitude of a normalized generator with unit internal impedance (pure resistance).

The generator produces a voltage  $V'_2$  across the one ohm resistance in the Darlington representation of the load as a reactance 2-port terminated in unit resistance as shown in Fig. MRI-13993. This voltage is related to  $V_1$  by the voltage scattering function  $S_{12}(p)$  of the overall network (Fig. MRI-13993). Thus

$$V'_2 = S_{12}(p) \frac{V_1}{2} \quad (8)$$

Since the power at the input to the load is the same as that delivered to the one ohm resistor in the Darlington representation

$$|V_2|^2 g(\omega) = |V'_2|^2 \quad (9)$$

where  $g(\omega)$  is the input conductance of the load.

Combining equations (7), (8) and (9)

$$e^2 = \left| \frac{V_2}{V_1} \right|^2 = \frac{|S_{12}(j\omega)|^2}{4 g(\omega)} \quad (10)$$

Since  $|S_{12}(j\omega)|^2 \leq 1$  it is immediately clear that in any physical network

$$Q^2 \leq \frac{1}{4g(\omega)} \quad (11)$$

In equation (10)  $g(\omega)$  is specified by the load alone and  $Q$  is directly proportional to  $|S_{12}(j\omega)|$ . Thus a necessary requirement for maximum voltage transfer is to maximize  $|S_{12}(j\omega)|$  consistent with the general theorem on realizability given earlier. This is accomplished precisely as in the power transfer problem when  $|S_{12}(j\omega)|$  and  $|S_{11}(j\omega)|$  (prescribed) are related to  $|S_{22}(j\omega)|$  by equations (3), (4), and (5). The general equation for optimum voltage transfer may then be written as

$$Q^2 = \frac{(1 - |S_{22}(j\omega)|)(1 + |S_{11}(j\omega)|)}{4g(\omega)} \quad (12)$$

$$|S_{22}(j\omega)| \geq |S_{11}(j\omega)|$$

The lossless and matched cases are then given as

$$Q^2 = \frac{1 - |S_{22}(j\omega)|^2}{4g(\omega)} \quad (\text{lossless}) \quad (13)$$

$$Q^2 = \frac{1 - |S_{22}(j\omega)|}{4g(\omega)} \quad (\text{matched}) \quad (14)$$

The integral equations for the latter two cases using equation (1) are:

$$\frac{1}{2} \int_0^\infty f_1(\omega) \ln \left( \frac{1}{1 - 4Q^2(\omega)g(\omega)} \right) d\omega = K_1 \quad (\text{lossless}) \quad (15)$$

$$\int_0^\infty f_1(\omega) \ln \left( \frac{1}{1 - 4Q^2(\omega)g(\omega)} \right) d\omega = K_1 \quad (\text{matched}) \quad (16)$$

The weighting functions  $f_1(\omega)$  and the parameters  $K_1$  are those tabulated in reference 3.

The only difference in form for the integral constraints in the two special cases is the factor  $1/2$ . However, since  $Q$  appears under the

integral sign the solution for maximum scale factor is generally formed from a transcendental equation so that there is no direct relation between the voltage gain of lossless and matched equalizers even in the flat transfer case. In this latter case

$$Q^2(\omega) = \begin{cases} C^2 & \omega_1 \leq \omega \leq \omega_2 \\ 0 & 0 \leq \omega < \omega_1, \omega > \omega_2 \end{cases} \quad (17)$$

where  $C$  is the voltage gain constant to be maximized. The integrals (15) and (16) are then

$$\frac{1}{2} \int_{\omega_1}^{\omega_2} f_1(\omega) \ln \left( \frac{1}{1 - 4C^2 g(\omega)} \right) d\omega = K_1 \quad (\text{lossless}) \quad (18)$$

$$\int_{\omega_1}^{\omega_2} f_1(\omega) \ln \left( \frac{1}{1 - 4C^2 g(\omega)} \right) d\omega = K_1 \quad (\text{matched}) \quad (19)$$

In effect the problem of a flat voltage equalizer reduces to the solution of a power transfer problem where a non-flat power gain curve shape is specified.

#### B. Example - Flat Voltage Equalizer for R-L Load

As an example of a voltage equalizer problem consider the case of a load consisting of the series combination of coil  $L$  and resistor  $R$ . The voltage transfer characteristic is to be a high pass one specified by:

$$Q(x) = \begin{cases} C & x_c \leq x \leq \infty \\ 0 & 0 \leq x < x_c \end{cases} \quad (20)$$

where  $x$  is a normalized frequency variable and  $x_c$  is its cut off value:

$$x = \omega \frac{L}{R} \quad (21)$$

$$x_c = \omega_c \frac{L}{R} \quad (22)$$

The load conductance is

$$g(x) = \frac{1}{R} \frac{1}{1 + x^2} \quad (23)$$

Since the load has only a simple zero of transmission at infinity the weighting function  $f(\omega)$  is unity and the integration constant  $K_1$  is given by<sup>3</sup>,

$$K \leq \frac{\pi R}{L} \quad (24)$$

The possibility of attaining the equal sign in equation (24) is dictated by equation (11). If equation (23) is substituted in that equation then an upper bound is set on  $Q^2(x)$  for any value of  $x$ :

$$Q^2(x) \leq \frac{R(1+x^2)}{4} \quad (25)$$

The permissible value of voltage gain increases with  $x$ , and thus for flat response

$$C^2 \leq \frac{R(1+x_0^2)}{4} \quad (26)$$

Since the lowest permissible gain occurs at cut-off. The upper limit for  $K = \pi R/L$  can only be attained if the value of  $C$  required in the integral equations does not violate equation (26). The integral relations given by equation (18) and (19) become:

$$A \frac{R}{L} \int_{x_c}^{\infty} \ln \frac{x^2 + 1}{x^2 + a^2} dx \leq \frac{\pi R}{L} \quad (27)$$

where

$$A = \begin{cases} 1 & \text{Matched case} \\ \frac{1}{2} & \text{Lossless case} \end{cases} \quad (28)$$

$$\text{and } a^2 = 1 - \frac{4C^2}{R} \quad (29)$$

For  $a^2 \geq 0$  integration of equation (27) gives

$$\pi(1-a) - x_c \ln \frac{1+x_c^2}{a^2+x_c^2} - 2 \tan^{-1} x_c + 2a \tan^{-1} \frac{x_c}{a} = \pi/A, \quad a^2 \geq 0 \quad (30)$$

The equal sign is used in order to determine whether the value of  $a$  (hence  $C$  by equation (29)) exceeds the limit of equation (26). When  $A = 1$ , the only real solution for  $a$  in equation (30) occurs when  $x_c = 0$ . In that case:

$$a = a^2 = 0, \quad C = \frac{R}{2} \quad (31)$$

The limit of equation (26) for  $x_0 = 0$  is also  $C = R/2$ , so that for flat transfer over  $0 \leq x \leq \infty$ , the maximum value permitted by the integral constraint for a matched equalizer can be obtained and this flat gain is precisely the d.c. gain. A lossless equalizer would give no gain advantage, since the solution of equation (30) with  $A = 1/2$  results in a value of  $C$  exceeding that permitted by equation (26).

For values of  $x_0 > 0$ , negative values of  $a^2$  are required to satisfy equation (30). Under these conditions the transcendental equation becomes:

$$\pi - x_0 \cdot \ln \frac{x_0^2 + 1}{x_0^2 - b^2} + b \ln \frac{x_0 - b}{x_0 + b} + 2 \tan^{-1} x_0 = \frac{\pi}{A} \quad (32)$$

Where

$$b^2 = -a^2 = \frac{4C^2}{R} - 1 \quad (33)$$

For values of  $0 \leq x_0 \leq 1.9$ , the solution of equation (32) for  $C$  always exceeds that permitted by equation (26) for both values of  $A$  (lossless and matched cases). In this region, where the value of  $\omega L/R$  is small at cut-off, the optimum flat gain is given by equation (26)

$$C = \sqrt{\frac{R(1 + x_0^2)}{2}}, \quad 0 \leq x_0 \leq 1.9 \quad (34)$$

and the lossless equalizer gives no advantage in gain over the matched equalizer. For the medium range of  $1.9 < x_0 \leq 4.8$ , the flat gain of a matched equalizer as obtained from equation (32) is less than that given by equation (26), while the gain of the lossless equalizer is still limited by equation (26). Finally for the high range  $4.8 < x_0 \leq \infty$ , both lossless and matched equalizers have gains limited by the solution of equation (32).

When  $x_0$  is very large equation (32) is approximated very well by:

$$b^2 + 1 = 2 + \frac{\pi x_0}{A}, \quad x_0 \gg 1 \quad (35)$$

Using equation (33), the optimum gain in this case is

$$\left. \begin{aligned} C &= \frac{\sqrt{R}}{2} \sqrt{2 + \frac{\pi x_0}{A}} \\ &\approx \frac{1}{2} \frac{\pi L}{A} \omega_c \end{aligned} \right\} \quad x_0 = \frac{\omega_c L}{R} \gg 1 \quad (36)$$

Thus as the load resistance becomes negligible compared to the load reactance at cutoff, the ratio of maximum flat gain of lossless ( $A = 1/2$ )

and matched ( $A = 1$ ) equalizers becomes:

$$\frac{C(\text{lossless})}{C(\text{matched})} = \sqrt{2} \quad (37)$$

The comparison of performance of lossless and matched equalizers for a flat high pass gain characteristic is summarized in graphical form in Fig. MRI-13994. The heavy line is the bounding curve defined by equation (29), and the dashed curves show the maximum flat gain of the lossless and matched equalizers as prescribed by the integral constraints.

#### V. Voltage Transfer Equalization of Reactive Load

The case of optimum voltage transfer from a finite generator to a reactive load is an important practical problem. However, the basic realizability theorem quoted in Section 2 is not readily applicable. Accordingly this equalization problem will be treated in a somewhat different way through the general point of view outlined in Section 1 will still be used. The optimum voltage equalization of an arbitrary lossless termination has not been considered in any complete fashion elsewhere, so that some details of the derivation of the realizability criteria (general constraints plus load constraints) will be given here. This will also serve to further illuminate the basis for the realizability theorem 1 of Section 2 since the two derivations parallel each other. It will be seen that whereas the scattering coefficients were a natural tool for handling the equalization of a dissipative load, the open circuit impedance elements are more directly applicable to the reactive load problem.

In Fig. MRI-13995a a finite generator of voltage  $V_1$  and unit internal resistance is shown driving an equalizer terminated in an arbitrary reactance. The amplitude ratio of output load voltage to open circuit generator voltage is

$$\rho = \left| \frac{V_2}{V_1} \right| = \left| \bar{Z}_{12}(\omega) \right| \quad (\text{general}) \quad (38a)$$

where  $|\bar{Z}_{12}(\omega)|$  is the amplitude of open circuit transfer impedance of an overall network shown in Fig. MRI-13995b, consisting of the equalizer shunted at the generator side by a 1 ohm resistor and at the output side by the reactive load.

If the equalizer is designed to produce an input match, then the voltage ratio may be written as:

$$\rho = \left| \frac{V_2}{V_1} \right| = \frac{1}{2} \left| Z_{12}(\omega) \right| \quad (\text{matched equalizer}) \quad (38)$$

where  $|Z_{12}(\omega)|$  is the amplitude of the open circuit transfer impedance of the matching equalizer plus lossless load. (Generator resistance is not included.)

If the equalizer is lossless then in Fig. MRI-13995b with excitation at the load side of the overall network, the power delivered here is equal to that dissipated in the 1 ohm shunting resistor. Thus

$$Q^2 = |\bar{Z}_{21}(\omega)|^2 = |\bar{Z}_{12}(\omega)|^2 = \bar{R}_{22}(\omega) \quad (\text{lossless equalizer}) \quad (39)$$

where  $\bar{R}_{22}(\omega)$  is the real part of the open circuit output driving point impedance of the overall network on  $j\omega$ :

$$\bar{R}_{22}(\omega) = \operatorname{Re} \bar{Z}_{22}(j\omega) \quad (40)$$

Since the equalization functions to be optimized (equations (38), (39)) involve the open circuit impedance parameters, the requirements on overall network physical realizability will be expressed in terms of these parameters. The necessary and sufficient conditions that an open circuit impedance matrix correspond to a physical 2-port is that the matrix be positive real<sup>14</sup>. That is

$$a) \quad R_{11}(\omega) R_{22}(\omega) - R_{12}^2(\omega) \geq 0, \quad R_{11}(\omega) \geq 0 \quad (41)$$

b) The open circuit impedances have no poles in the right half plane.

c) Impedance element poles on the boundary be simple and the residues of  $Z_{12}(p)$ ,  $Z_{11}(p)$  and  $Z_{22}(p)$  at these poles satisfy

$$a_{11}a_{22} - a_{12}^2 \geq 0 \quad a_{11} \geq 0$$

Equation (41) is the major physical constraint in the equalizer problem since if this is satisfied the remaining two requirements can in general be met by suitable design of the actual equalizer network.

The constraints of the load can be easily established by observing that the reactive load impedance  $Z_L(p)$  is in parallel with the impedance seen looking in at the back of the equalizer. Thus

$$Z_{22} = \frac{z_{22} Z_L}{z_{22} + Z_L} \quad (42)$$

$$\bar{Z}_{22} = \frac{\bar{z}_{22} Z_L}{\bar{z}_{22} + Z_L} \quad (43)$$

where  $Z_{22}$  is the back end open circuit driving point impedance for the



equalizer plus load with the 1 ohm resistor at the input removed, and  $\bar{Z}_{22}$  is a similar quantity for the overall network of Fig. MRI-13995b. ( $z_{22}$  is back end equalizer impedance)

Any zero of the reactive load must be simple and occur on  $j\omega$ . Further if  $Z_L$  is expanded in a power series at this zero, the first non vanishing Taylor coefficient is positive.\* Inspection of equation (42) and (43) show that the back end impedance  $Z_{22}$  or  $\bar{Z}_{22}$  must vanish at this point and in the vicinity of the zero,  $p_1 = j\omega_1$ :

$$Z_{22}(p) = \bar{Z}_{22}(p) = Z_L(p) = a_1(p-p_1) \quad p \rightarrow p_1 \quad (44)$$

Equation (27) is in fact entirely independent of the equalizer and its input termination.

The requirement that the back end impedance of the overall network be constrained by equation (27) can be expressed in integral form by using the Cauchy formulas for Taylor coefficients. These take the form in the present instance of:

$$\frac{1}{2\pi j} \oint g_1(p) Z_{22}(p) dp = a_1 \quad (45)$$

where the closed path of integration is along  $j\omega$  (avoiding by small semi-circular indentations any boundary poles) and is completed around the semi-circle of infinite radius enclosing the right half  $p$  plane.  $g_1(p)$  are the weighting functions to select the appropriate Taylor coefficient of the load,  $a_1$ . If the contributions of the small indentations are accounted for, and  $g_1(p)$  is modified slightly to be even in  $\omega$  along  $j\omega$ , equation (45) may be written in terms of  $R_{22}(\omega)$  as follows:

$$\int_0^{\infty} \frac{1}{\omega^2} R_{22}(\omega) d\omega = \frac{\pi a_0}{2} - \pi \sum_m \frac{b_m}{\omega_m^2} \quad (\text{zero at } p = 0) \quad (46)$$

$$\int_0^{\infty} R_{22}(\omega) d\omega = \frac{\pi a_{\infty}}{2} - \pi \sum_m b_m \quad (\text{zero at } p = \infty) \quad (47)$$

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\* This follows from Foster's reactance theorem.<sup>15</sup>

$$\int_0^{\infty} \frac{1}{(\omega^2 - \omega_1^2)^2} R_{22}(\omega) d\omega = \frac{\pi a_1}{4 \omega_1^2} - \pi \sum_m \frac{b_m}{(\omega_m^2 - \omega_1^2)^2} \quad (\text{zero at } p = \pm j\omega_1) \quad (48)$$

$\omega_m \neq \omega_1$

In every case  $b_m$  is the residue of any  $Z_{22}$  poles on  $j\omega$  and is always positive. Such poles can only reduce the permissible integration constant. The only other possibility of reducing the right hand sides of equation (46), (47), (48) is for  $z_{22}$  (or  $\bar{z}_{22}$ ) of the equalizer to have a zero coincident with  $Z_L$ . Since these relations must all be satisfied simultaneously by  $Z_{22}$  (or  $\bar{Z}_{22}$ ), it may be necessary to introduce elements in the equalizer which cause coincident zeros and/or additional poles on  $j\omega$ . Another point which should be mentioned here is that referring to equation (44), equations (45), (46), (47), (48) are valid both for the barred and unbarred quantities.

Finally since any zero of the load is a zero of voltage transfer, the following physical realizability theorem may be stated:

#### Theorem 2

The necessary and sufficient conditions that a combined network consisting of a two port and a prescribed reactive termination be physically realizable is

- a) The combined network must have a positive real open circuit impedance matrix.
- b) All zeros of the reactive load impedance must be contained in the open circuit transfer impedance of the combined network. (The zeros are all simple.)
- c) The integral constraints given by equation (46), (47), (48) and summarized by the form

$$\oint_0^{\infty} f_1(\omega) R_{22}(\omega) d\omega = K_1 \quad (49)$$

must be simultaneously satisfied by the back end resistance of the combined network at each zero of the load.

The similarity between this theorem and theorem 1 is obvious.

## VI. Flat Gain Equalization of a Reactive Load

### A. Form of Integral Constraints

As an example of the application of theorem 2, the case of flat voltage transfer for a prototype low pass equalizer defined by:

$$\rho = \left| \frac{V_2}{V_1} \right| = |\bar{Z}_{12}(\omega)| = \begin{cases} \bar{C} & \text{for } 0 \leq \omega \leq 1 \\ 0 & \text{for } 1 < \omega \leq \infty \end{cases} \quad (50)$$

will be examined.

In the case of the matched equalizer, condition a of theorem 2 leads to

$$R_{22}(\omega) - R_{12}^2(\omega) \geq 0 \quad (\text{Matched equalizer}) \quad (51)$$

This is obtained from equation (41) with  $R_{11}(\omega) = 1$ ,  $0 \leq \omega < \infty$ .

Now

$$R_{12}(\omega) = |Z_{12}(\omega)| \cos(\varphi(\omega) + \theta(\omega)) \quad (52)$$

Where  $\varphi$  is the minimum phase characteristic of the combined network (equalizer and load), and  $\theta(\omega)$  any phase characteristic obtained with a tandem combination of unit voltage transfer networks, considered as part of the equalizer (i.e. the latter need not be a minimum phase network).

The function  $\varphi(\omega)$  is directly found graphically or analytically from the shape of the  $Z_{12}(\omega)$  amplitude function and is not affected by the scale factor on this function.<sup>1</sup>  $\theta(\omega)$  is independent of  $\varphi(\omega)$  and is continuous with positive slope.<sup>1</sup> If equation (52) is substituted into (51), it is clear that a necessary condition for maximum  $|Z_{12}(\omega)|$  is that the equal sign be used. Thus

$$R_{22}(\omega) = |Z_{12}(\omega)|^2 \cos^2(\varphi + \theta) = R_{12}^2(\omega) \quad (53)$$

This means that the combined matched equalizer-load network will require no more than 1 resistor.<sup>14</sup> Suppose that  $|Z_{12}(\omega)|$  is represented as

$$|Z_{12}(\omega)| = \begin{cases} C & 0 \leq \omega \leq 1 \\ \frac{C}{\omega^k} & 1 < \omega \leq \infty \end{cases} \quad (k \text{ a positive integer}) \quad (54)$$

where

$$C = 2 \bar{C} \quad (55)$$

As  $k \rightarrow \infty$ , the transfer characteristic defined by equation (54) approaches the gain shape given by equation (50) (it differs by the factor  $1/2$ ).

The minimum phase characteristic defined by equation (54) is given below: (See reference 1, pages 312-318)

$$\varphi(\omega) = \frac{k}{\pi} \int_1^{\infty} \log \coth \frac{|u|}{2} du = \begin{cases} k \varphi_1(\omega) & 0 \leq \omega \leq 1 \\ k \varphi_2(\omega) & 1 < \omega \leq \infty \end{cases} \quad (56)$$

In this equation  $u = \log \Omega/\omega$ , where  $\Omega$  is the running frequency variable of integration. The definite integral is of course a function of  $\omega$ .

The approximate values of  $\varphi_1(\omega)$  and  $\varphi_2(\omega)$  are given below for reference only. They are not used in the derivation, and merely indicate the nature of these functions. They are continuous and monotonic.

$$\varphi_1(\omega) \cong -k \frac{2\omega}{\pi} \quad 0 \leq \omega \leq 1 \quad (56a)$$

$$\varphi_2(\omega) \cong -k \frac{\pi}{2} \quad 1 < \omega \leq \infty \quad (56b)$$

The integral equation (49) of theorem 2 for the matched flat voltage equalizer becomes

$$C^2 \int_0^1 f_1(\omega) \cos^2 [k\varphi_1(\omega) + \theta(\omega)] d\omega + C^2 \int_1^\infty \frac{f_1(\omega)}{\omega^k} \cos^2 (k\varphi_2(\omega) + \theta(\omega)) d\omega = K_1$$

The second integral clearly approaches zero as  $k$  becomes infinite. Expanding the squared cosine term, the first integral may be written as:

$$\begin{aligned} \frac{1}{2} \int_0^1 f_1(\omega) d\omega + \frac{1}{2} \int_0^1 [f_1(\omega) \cos 2\theta(\omega)] \cos 2k\varphi_1(\omega) d\omega \\ + \frac{1}{2} \int_0^1 [f_1(\omega) \sin 2\theta(\omega)] \sin 2k\varphi_1(\omega) d\omega \end{aligned}$$

Now let  $\gamma(\omega) = 2\varphi_1(\omega)$ , hence  $\gamma'(\omega) d\omega = d\gamma$ ,  $\omega = h(\gamma)$  so that the second and third integrals above become:

$$\frac{1}{2} \int_{\gamma(0)=0}^{\gamma(1)} g_1(\gamma) \cos k\gamma d\gamma + \frac{1}{2} \int_{\gamma(0)=0}^{\gamma(1)} g_2(\gamma) \sin k\gamma d\gamma$$

where

$$g_1(\gamma) = \frac{f_1(h(\gamma)) \cos 2\theta(h(\gamma))}{\gamma'(h(\gamma))}, \quad g_2(\gamma) = \frac{f_1(h(\gamma)) \sin 2\theta(h(\gamma))}{\gamma'(h(\gamma))}$$

The first of these integrals is essentially the  $k$ 'th order Fourier coefficient for a function defined equal to  $g_1(\gamma)$  in the interval  $\gamma(0)$  to  $\gamma(1)$ , periodic for all  $\gamma$  and even in  $\gamma$ . The second is a  $k$ 'th order Fourier coefficient for a function defined as equal to  $g_2(\gamma)$  over the interval  $\gamma(0)$  to  $\gamma(1)$ , periodic and odd in  $\gamma$ . As  $k$  becomes large, the Fourier coefficients approach zero, hence these integrals vanish in the limit  $k \rightarrow \infty$ .

The final representation of the integral constraint for a matched equalizer with the low pass flat voltage gain of equation (50) is therefore

$$2 \bar{C}^2 \int_0^1 f_1(\omega) d\omega = K_1 \text{ (matched flat low pass case)} \quad (57)$$

A similar derivation for a flat band pass over the interval  $\omega_1$  to  $\omega_2$  (zero gain elsewhere) gives

$$2 \bar{C}^2 \int_{\omega_1}^{\omega_2} f_1(\omega) d\omega = K_1 \text{ (matched flat band pass case)} \quad (57a)$$

Both these equations (57) and (57a) are entirely independent of whether minimum or non-minimum phase networks are used, but apply only to a flat voltage gain response of amplitude  $\bar{C}$  over a finite interval with zero gain elsewhere.

Returning to the low pass case, let

$$\int_0^1 f_1(\omega) d\omega = A_1 \quad (58)$$

equation (57) may then be written

$$\bar{C}^2 = \frac{K_1}{2A_1} \text{ (Matched)} \quad (59)$$

This equation is independent of whether minimum or non-minimum phase networks are used.

For a lossless equalizer, equation (39) may be used directly in connection with equation (50) and placed into the expression of theorem 2. Thus the lossless equalizer with flat transfer characteristics given by equation (50) must satisfy:

$$\int_0^\infty f_1(\omega) \bar{R}_{22}(\omega) d\omega = \int_0^\infty f_1(\omega) |\bar{Z}_{12}(\omega)|^2 d\omega = \bar{C}^2 \int_0^1 f_1(\omega) d\omega \quad (60)$$

or

$$\bar{C}^2 = \frac{K_1}{A_1} \text{ (Lossless case)} \quad (61)$$

and the ratio of optimum matched and lossless equalizer gains for the flat transfer case and reactive load is:

$$\frac{\bar{C} \text{ (lossless)}}{\bar{C} \text{ (matched)}} = \sqrt{2} \quad (62)$$

### B. Example of a C-LC Load

As an example of the application of the material given above suppose it is required to design a flat voltage equalizer over the band  $0 \leq \omega \leq 1$  for a three element load. The load consists of a capacitor  $C'$  in shunt with a series circuit of coil  $L$  and condenser  $C$ .

This load impedance has zeros at  $p = \infty$ , and  $p = \pm j\omega_1$  with  $\omega_1 = 1/LC$ .\*

The Taylor coefficients at these zeros are:

$$(p = \infty) a_{\infty} = \frac{1}{C'} \quad (63)$$

$$(p = \pm j\omega_1) a_1 = \frac{2L}{C} \quad (64)$$

Referring to equations (47) and (48) and neglecting the pole residue terms:

$$K_1 = \frac{\pi a_{\infty}}{2} = \frac{\pi}{2C'} \quad (65)$$

$$K_2 = \frac{\pi a_1}{4 \omega_1^2} = \frac{\pi}{2} L^2 \quad (66)$$

For the zero at infinity the weighting function is unity and

$$A_1 = \int_0^1 d\omega = 1 \quad (67)$$

For the zero at  $\pm j\omega_1$ , the weighting function is  $f_2(\omega) =$

$$\frac{1}{(\omega^2 - \omega_1^2)^2}$$

and

$$A_2 = \int_0^1 \frac{d\omega}{(\omega^2 - \omega_1^2)^2} = \frac{1}{2\omega_1^2(\omega_1^2 - 1)} + \frac{1}{4\omega_1^3} \ln \frac{\omega_1 + 1}{\omega_1 - 1} \quad (68)$$

---

\* Presumed outside of pass-band.

Consider a specific numerical case with

$$C' = \frac{1}{2} \quad (69)$$

$$L = C = \frac{1}{\sqrt{2}}$$

For these conditions the various constants may be evaluated to give:

$$\frac{K_1}{A_1} = \pi \quad (70)$$

$$\frac{K_2}{A_2} = \frac{\pi/4}{0.35} = \frac{\pi}{1.40} \quad (71)$$

The only way to satisfy these constraints simultaneously is to reduce the value of  $K_1$  by using parasitic elements in the equalizer. This can be done for example by use of a shunt condenser across the equalizer output and in parallel with the load of value  $C'' = 0.2$ . As a result the maximum gain of this equalizer is not limited by the capacitance  $C'$ , but by the series LC circuit and the maximum value of this gain for a low pass matched equalizer is

$$\bar{C} = \sqrt{\frac{\pi}{2.80}} \quad (\text{matched}) \quad (72)$$

and for a lossless equalizer the maximum gain is

$$\bar{C} = \sqrt{\frac{\pi}{1.40}} \quad (\text{lossless}) \quad (73)$$

It is up to the designer to decide whether a matched input to the equalizer system is worth sacrificing for a 40% increase in gain.

### C. Design of a Finite Equalizer Network

In the previous discussion ideal characteristics were assumed for the equalizer amplitude response. The voltage gain was presumed absolutely flat in the pass band, and the cut-off was taken as infinitely sharp. In order to apply theorem 2 to the design of a finite network, a gain characteristic may be assumed in analytic form (i.e. an even rational function of  $\omega$  which is always positive). The analysis in such a case involves determination of the  $R_{22}$  characteristic and its substitution in the integral equations to determine the maximum gain scale factor. To illustrate this procedure a simple example will be given for the voltage equalization and

match of a unit capacitive termination\*. A Butterworth type of low pass response is assumed for a matched equalizer

$$|Z_{12}(\omega)|^2 = \frac{C^2}{1 + \omega^4} \quad (74)$$

The amplitude of  $Z_{12}(\omega)^2$  is down by 1/2 at  $\omega = 1$  which is considered the normalized cut-off frequency. C is a gain constant whose maximum value is to be determined by applying theorem 2.

Equation (74) may be factored to give the open circuit transfer impedance  $Z_{12}(p)$  of the required combined matched equalizer-load network. This function must have no right half plane poles.

$$Z_{12}(p) = \frac{C}{p^2 + \sqrt{2} p + 1} \quad (75)$$

The impedance  $Z_{12}(p)$  has the zero of the load at infinity as prescribed by theorem 2 because of the particular choice of amplitude function in equation (74).

The real component of  $Z_{12}(p)$  along  $p = j\omega$  is

$$R_{12}(\omega) = C \frac{1 - \omega^2}{1 + \omega^2} \quad (76)$$

and applying equation (53) for the optimum matched equalizer

$$R_{22}(\omega) = R_{12}^2(\omega) = C^2 \frac{(1 - \omega^2)^2}{(1 + \omega^2)^2} \quad (77)$$

The complex impedance  $Z_{22}(p)$  may be obtained by expanding  $R_{22}(\omega)$  in partial fractions as described on pages 204-205 of reference 1. The result\*\* is

$$Z_{22}(p) = C^2 \frac{0.363 p^5 + p^2 + 1.066 p + 1}{p^4 + 2.82 p^3 + 4p^2 + 2.62 p + 1} \quad (78)$$

This function has the load zero at infinity as prescribed by theorem 2.

The integral constraint (equation (47)) gives for unit capacitive load:

\* This example is taken from a thesis by L. Norde for the degree M.E.E. at the Polytechnic Institute of Brooklyn, June 1953.

\*\* In general if the resulting  $Z_{22}(p)$  function does not contain the required load zeros, it is necessary to form new rational functions for  $|Z_{12}(\omega)|^2$  (possibly by the use of common numerator and denominator factors) until this requirement is satisfied.



$$\int_0^{\infty} R_{22} d\omega = C^2 \int_0^{\infty} \frac{(1 - \omega^2)^2}{(1 + \omega^4)^2} d\omega = \frac{\pi}{2} \quad (79)$$

If the definite integral is evaluated, the gain factor  $C$  is found to be

$$C = 1.68 \quad (80)$$

or

$$\bar{C} = \frac{C}{2} = 0.84 \text{ (matched case)} \quad (81)$$

The maximum gain for the completely ideal low pass characteristic (in the matched case) obtained from equation (59) is

$$\bar{C} \text{ (ideal)} = 0.89 \text{ (matched case)} \quad (82)$$

so that the simple characteristic of equation (74) is a reasonable compromise.

The final matched equalizer plus load is specified by equations (75), (78) and

$$Z_{11}(p) = 1$$

The resultant network is shown in Fig. MRI-13996a. Observe that the load capacitance is removable leaving a physical equalizer, and only 1 resistor appears in the network.

The design of a lossless equalizer from a specified  $|\bar{Z}_{12}(\omega)|^2$  characteristic which satisfies theorem 2 is straightforward. The equalizer network is merely synthesized from the back end as a reactance 2-port terminated in a 1 ohm resistor by the Darlington procedure for the special case of an infinite impedance generator.<sup>10</sup> The 1 ohm resistor (input generator impedance) and load are then removed, and the remaining lossless network is the required equalizer.

For the example of equation (74) with

$$|\bar{Z}_{12}(\omega)|^2 = \frac{\bar{C}^2}{1 + \omega^4} = R_{22}(\omega) \quad (83)$$

The integral requirement gives

$$\int_0^{\infty} \frac{\bar{C}^2}{1 + \omega^4} d\omega = \frac{\pi}{2}$$

and

$$\bar{C} = 2^{0.25} = 1.19 \text{ (lossless equalizer)} \quad (84)$$

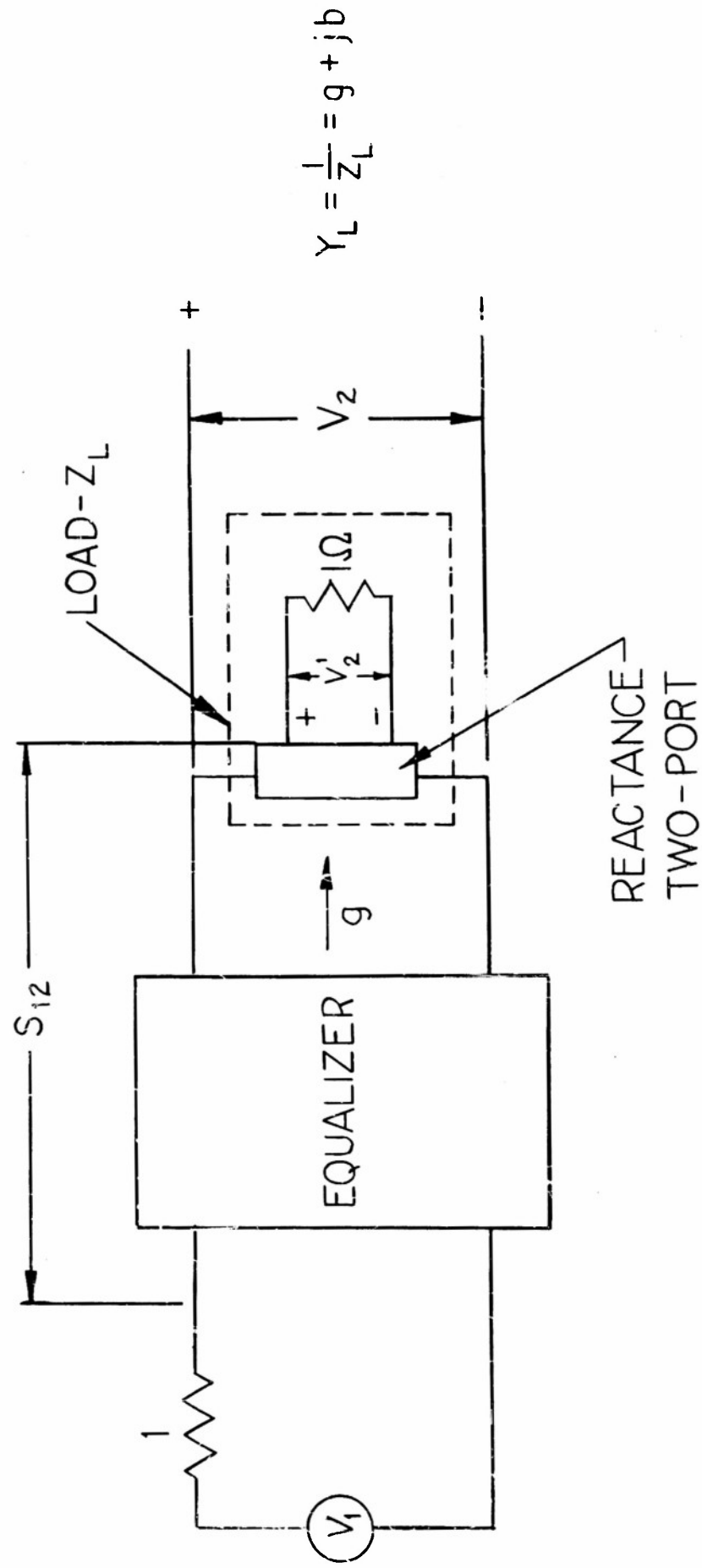
The network is shown in Fig. MRI-13996b.

\* This constraint could have been directly obtained from the network synthesis by choosing  $\bar{C}$  to give the prescribed load capacitance.

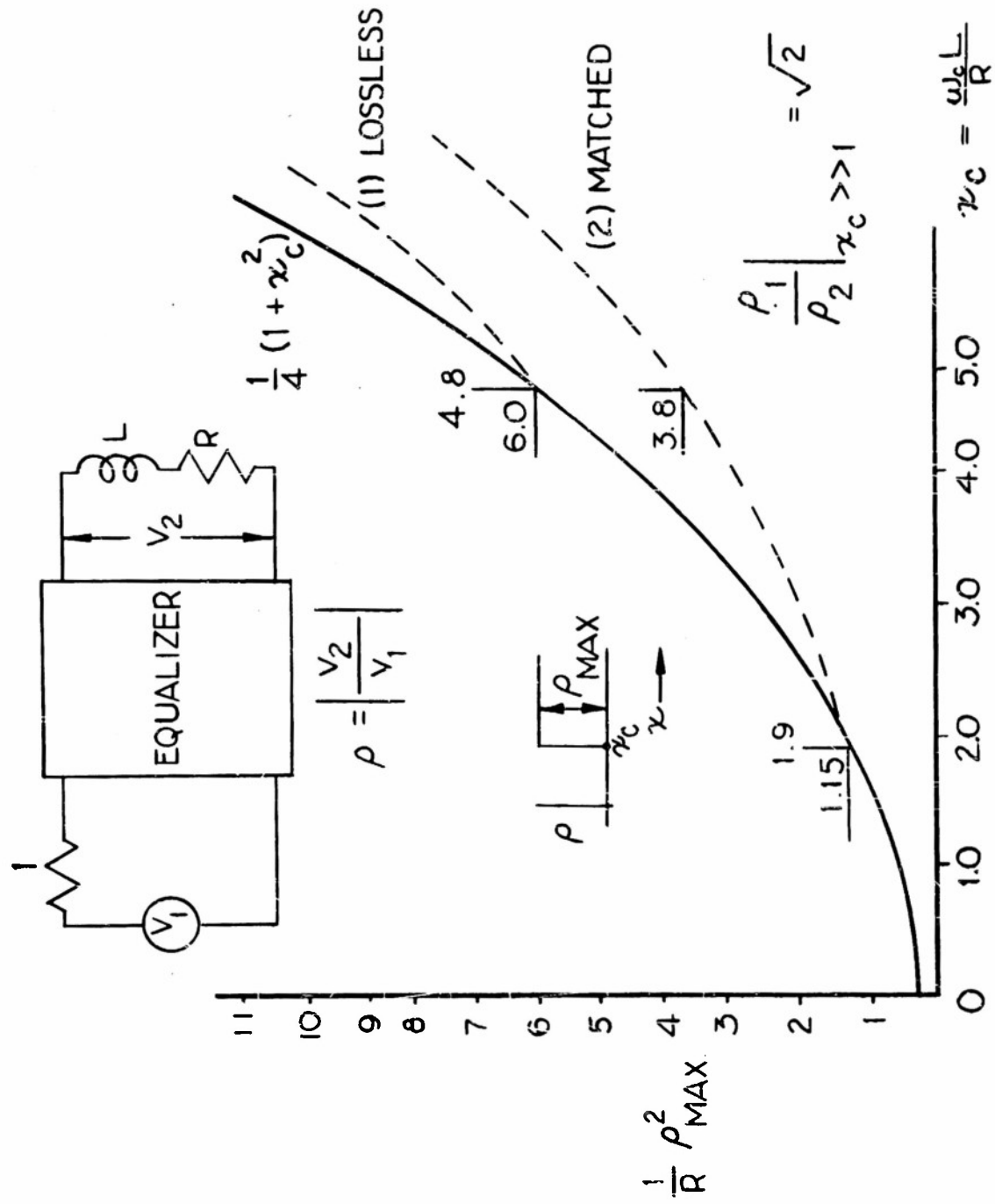
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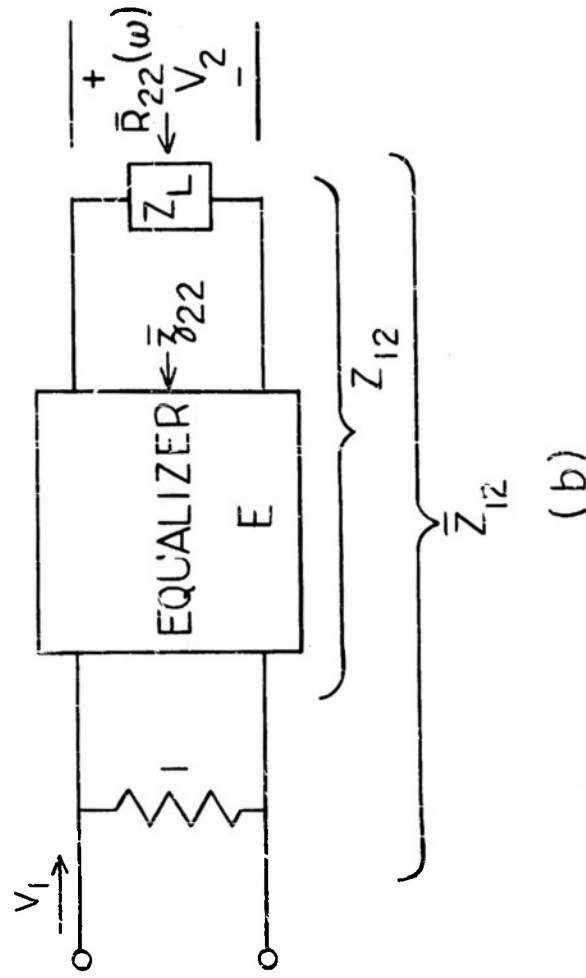
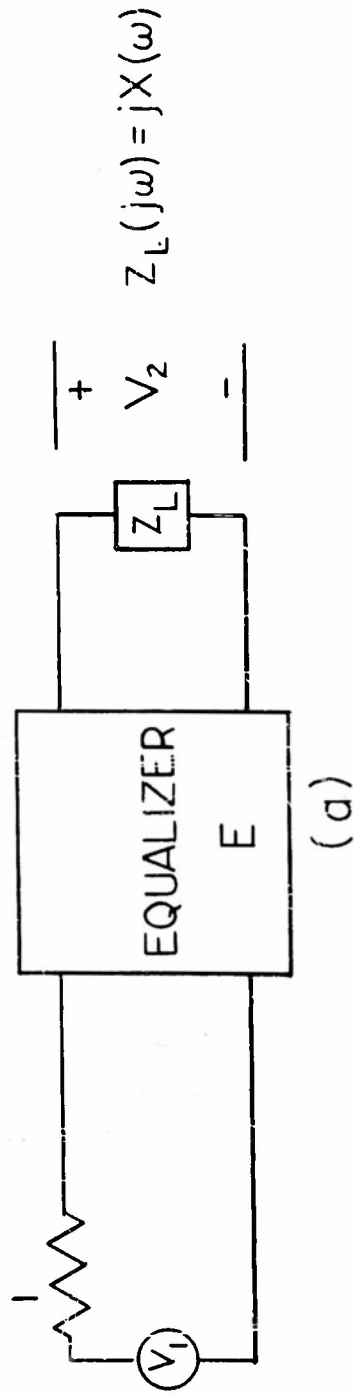
# VOLTAGE TRANSFER - GENERAL LOAD



# VOLTAGE EQUALIZER R-L LOAD

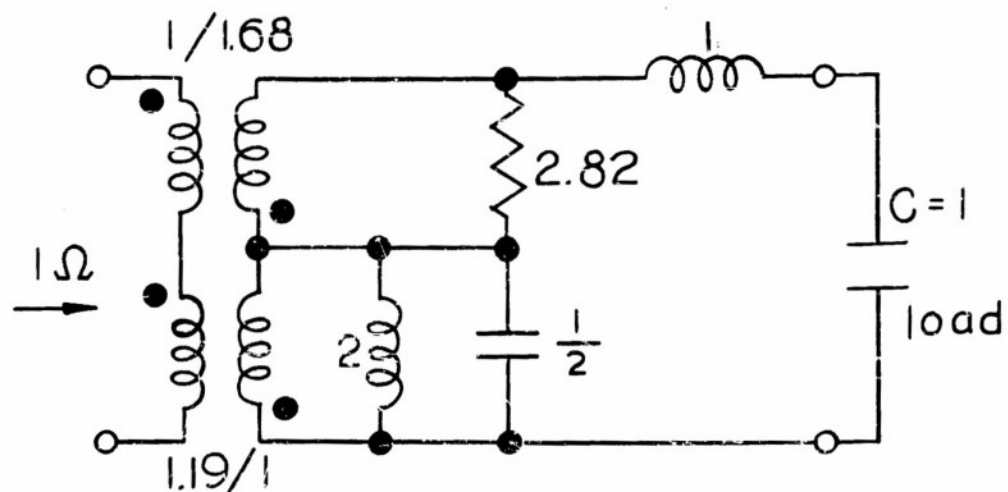


# VOLTAGE TRANSFER - REACTIVE LOAD

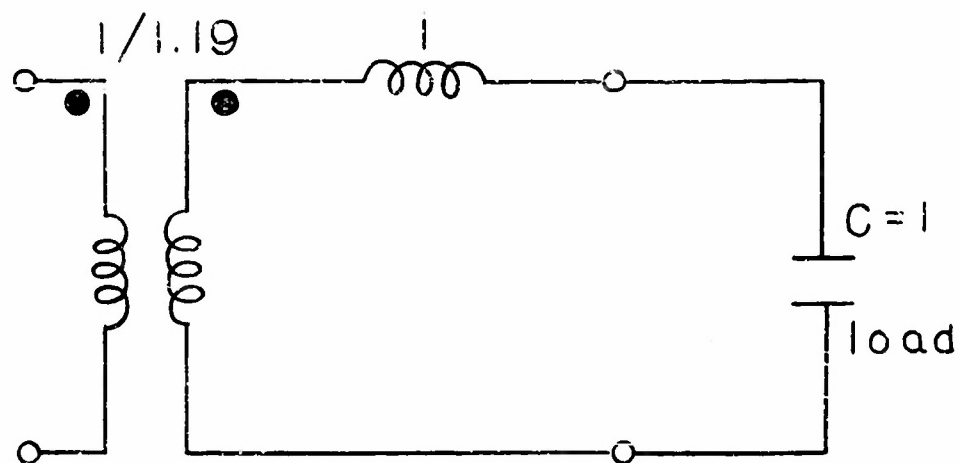


$$\frac{V_1}{V_2} = \bar{Z}_{12} = \frac{1}{2} Z_{12} \text{ (MATCHED CASE)}$$

$$|\bar{Z}_{12}|^2 = \bar{R}_{22} \text{ IF } E \text{ LOSSLESS}$$



(a) MATCHED EQUALIZER  $A = 0.84$



(b) LOSSLESS EQUALIZER  $A = 1.19$

EQUALIZERS FOR CAPACITIVE LOAD

$$\text{AND VOLTAGE GAIN} = \frac{A}{\sqrt{1 + \omega^4}}$$